I. Oscillations & Waves

Types of oscillators:

SHO → simple

DHO → damped

FHO → forced.

> Damped Harmonic Oscillator:

Damping → Decrease in amplitude.

Theory:

2-types of forces mainly

↓

Restoring force $(-\mathcal{C}x)$

Damping force $\left(-\gamma \frac{dx}{dt}\right)$

Force eq. of DHO:

$$\underset{\substack{\uparrow \\ (m \cdot a)}}{m\frac{d^2x}{dt^2}} = -\mathcal{C}x - \gamma\frac{dx}{dt}$$

γ = damping coeff

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \mathcal{C}x = 0 \qquad \div m$$

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{\mathcal{C}}{m}x = 0$$

$\frac{\gamma}{2m} = K$
↓
Damping const.

$\sqrt{\mathcal{C}/m} = \omega_0$
↓
natural angular freq.

Force eq.

$$\therefore \boxed{\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega_0^2 x = 0} \quad -① $$

Sol: $x = Ae^{\alpha t}$

$$\frac{dx}{dt} = A\alpha e^{\alpha t}$$

$$\frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

$$\therefore ① \rightarrow A\alpha^2 e^{\alpha t} + 2KA\alpha e^{\alpha t} + \omega_0^2 Ae^{\alpha t} = 0$$

$$Ae^{\alpha t}\left(\alpha^2 + 2K\alpha + \omega_0^2\right) = 0$$

$$\therefore \alpha^2 + 2K\alpha + \omega_0^2 = 0$$

$$\alpha = \frac{-2K \pm \sqrt{4K^2 - 4\omega_0^2}}{2}$$

$$\alpha_1 = -K + \sqrt{K^2 - \omega_0^2}$$

$$\alpha_2 = -K - \sqrt{K^2 - \omega_0^2}$$

Gen. sol. of DHO

$$x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

$$= A_1 e^{\left(-K + \sqrt{K^2 - \omega_0^2}\right)t} + A_2 e^{\left(-K - \sqrt{K^2 - \omega_0^2}\right)t}$$

$$x = e^{-Kt}\left[A_1 e^{\sqrt{K^2 - \omega_0^2}\,t} + A_2 e^{-\sqrt{K^2 - \omega_0^2}\,t}\right] \quad -②$$

Based on the values of K & ω_0, DHO can be classified into 3.

> $K > \omega_0 \rightarrow$ Over damped

> $K = \omega_0 \rightarrow$ Critically damped

> $K < \omega_0 \rightarrow$ Under damped

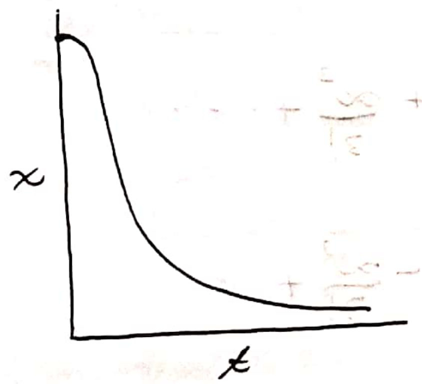
Case 1; $K > \omega_0$; Over damped.

Let $\sqrt{K^2 - \omega_0^2} = \beta$.

$$\therefore ② \rightarrow x = e^{-Kt}\left[A_1 e^{\beta t} + A_2 e^{-\beta t}\right]$$

$$= A_1 e^{(-K+\beta)t} + A_2 e^{(-K-\beta)t}$$

ie: The disp. varies exponentially.



dies off exponentially

Non-oscillatory.

Aperiodic / dead beat oscillators $\rightarrow$ used in dead-beat galv. meter.

Case 2: $K = \omega_0$; Critically damped

$$② \Rightarrow x = e^{-Kt}\left[A_1 e^{0t} + A_2 e^{-0x}\right]$$

$$= e^{-Kt}\left[A_1 + A_2\right]$$

Let $A_1 + A_2 = B$

$$\therefore \underline{\underline{x = Be^{-Kt}}} \rightarrow$$ Only one ~~variable~~ const.

$\therefore$ It doesn't form sol. of 2^{nd} order diff. eq.

$\therefore$ Substitute $\sqrt{K_2 - \omega^2} = h$; h is a small qty.

$$\Rightarrow x = e^{-Kt}\left[A_1 e^{ht} + A_2 e^{-ht}\right]$$

Note: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\therefore e^{ht} = 1 + \frac{ht}{1!} + \frac{(ht)^2}{2!} + \cdots \approx 1 + ht$$

$$e^{-ht} = 1 - ht$$

$$\therefore x = e^{-Kt}\left[A_1(1+ht) + A_2(1-ht)\right]$$

$$= e^{-Kt}\left[A_1 + A_2 + (A_1 - A_2)\cdot ht\right]$$

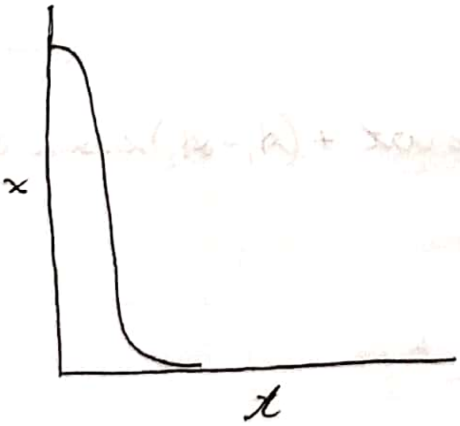
Let $A_1 + A_2 = D$

$$(A_1 - A_2)h = E$$

$$\therefore x = e^{-Kt}\left[D + Et\right]$$

$$\underline{\underline{x = (D + Et)\,e^{-Kt}}}$$

Due to the $(D+Et)$, there will be a max. value, then it decreases, faster than the over damped case.



[Non-oscillatory
Just aperiodic] → In galvanometers.

Case 3 : $K < \omega_0$; Under damped

$$\sqrt{K^2-\omega_0^2} = \sqrt{-(\omega_0^2-K^2)}$$

$$= i\omega \quad ; \quad \omega_0^2 - K^2 = \omega^2$$

ω = angular freq

$$\therefore x = e^{-Kt}\left[A_1 e^{\sqrt{K^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{K^2-\omega_0^2}\,t}\right]$$

$$= e^{-Kt}\left[A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\therefore x = e^{-Kt}\left[A_1(\cos\omega t + i\sin\omega t) + A_2(\cos\omega t - i\sin\omega t)\right]$$

$$= e^{-Kt}\left[(A_1+A_2)\cos\omega t + (A_1-A_2)\, i\sin\omega t\right]$$

Let $(A_1+A_2) = a_0\sin\phi$ &

$i(A_1-A_2) = a_0\cos\phi$

$$\therefore x = e^{-Kt}\left[a_0\sin\phi\cos\omega t + a_0\cos\phi\sin\omega t\right]$$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$.

$\therefore x = e^{-Kt} a_0 \sin(\phi + \omega t)$

$$\boxed{x = a_0 e^{-Kt} \sin(\omega t + \phi)}$$

where $\boxed{a_0 e^{-Kt}}$ → Amplitude of damped harmonic oscillators

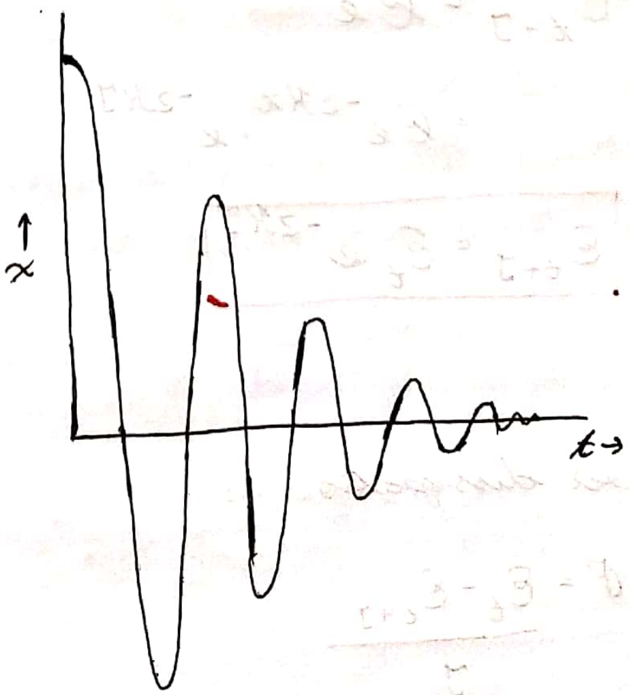
$\omega = \sqrt{\omega_0^2 - K^2}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - K^2}}$

In underdamped, two cases are possible.

> Decrease in ω, T increases

> Amplitude term decreases exponentially with time



Power dissipation of damped oscillator

It is the rate of dissipation of energy.

Defined as the ratio of energy loss in one period to the total time period

$$\boxed{P = \frac{\text{energy loss in 1 period}}{\text{tot. time period}}}$$

Tot. energy of oscillators

$$\mathcal{E}_t \propto \left(a_0 e^{-Kt}\right)^2$$

$$\boxed{\mathcal{E}_t = \mathcal{C} e^{-2Kt}} \longrightarrow \text{Tot. energy}$$

Energy of oscillator after 1 cycle/period

$$\mathcal{E}_{t+T} = \mathcal{C} e^{-2K(t+T)}$$

$$= \mathcal{C} e^{-2Kt} \cdot e^{-2KT}$$

$$\boxed{\mathcal{E}_{t+T} = \mathcal{E}_t e^{-2KT}}$$

$\therefore$ Power dissipation is:

$$P = \frac{\mathcal{E}_t - \mathcal{E}_{t+T}}{T}$$

$$= \frac{\mathcal{E}_t - \mathcal{E}_t e^{-2KT}}{T}$$

$$= \frac{\mathcal{E}_t - \mathcal{E}_t(1-2KT)}{T}$$; expn. of $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$

$$= \frac{\mathcal{E}_t - \mathcal{E}_t + 2KT\mathcal{E}_t}{T}$$

$$= \cancel{2K}\ \underline{\underline{\mathcal{E}_t 2K}}$$

$$\therefore \boxed{P = \frac{\mathcal{E}_t}{\tau}}$$; $\tau = \frac{1}{2K}$ → Relaxation time

↓

Time after which the energy is reduced to $1/e$ of its initial value.

Quality factor of damped oscillator (Q)

Quality factor is the ratio of 2π times the energy stored in the system to the energy lost/Energy dissipated per unit time.

i.e.: $$\boxed{Q = 2\pi \frac{\mathcal{E}_t}{PT}}$$

$$= 2\pi \frac{\cancel{\mathcal{E}_t}}{\left(\frac{\mathcal{E}_t}{\tau}\right)T}$$

$$\boxed{Q = \frac{2\pi}{T}\tau = \omega\tau}$$

Forced/Driven harmonic oscillator

An oscillator that oscillated by force at a particular frequency, other than its natural frequency is said to be FHO.
eg: Piano.

Theory

3-types of forces are experienced.

↓

Restoring force → $(-Cx)$

Damping force → $\left(-\gamma \frac{dx}{dt}\right)$

Ext. driving force → $(F_0 \sin pt)$ $p/2\pi \rightarrow$ freq

Force eq. of FHO

$$m\frac{d^2x}{dt^2} = -Cx - \gamma\frac{dx}{dx} + F_0 \sin pt$$

$$m\frac{d^2x}{dt^2} + Cx + \gamma\frac{dx}{dt} = F_0 \sin pt$$

$$\frac{d^2x}{dt^2} + \frac{C}{m}x + \frac{\gamma}{m}\frac{dx}{dt} = \frac{F_0}{m}\sin pt$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x + 2K\frac{dx}{dt} = \frac{F_0}{m}\sin pt$$

$\sqrt{C/m} = \omega_0$

$\frac{\gamma}{2m} = K$

$$\boxed{\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt} \quad \text{—(1)}$$

$\frac{F_0}{m} = f_0$

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Solution

It is a non-homogeneous diff. eq.

It contains 1) Complimentary funct (CF)

2) Particular integral (PI)

CF : $\cancel{\frac{d^2x}{}}$ $\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega_0^2 x = 0$

Its sol. is: $\boxed{x = a_0 e^{-kt} \sin(\omega t + \phi)}$

PI : Let $\boxed{x = A\sin(pt - \theta)}$

$\cancel{\frac{dx}{dt} = pA\sin(pt-\theta}$

$\frac{dx}{dt} = pA\cos(pt - \theta)$

$\frac{d^2x}{dt^2} = -p^2 A\sin(pt - \theta)$

$\therefore$ ① $\rightarrow$ $-p^2 A\sin(pt-\theta) + 2KpA\cos(pt-\theta)$

$+ \omega_0^2 A\sin(pt-\theta) = f_0 \sin pt$

$A\sin(pt-\theta)\left[\omega_0^2 - p^2\right] + 2KpA\cos(pt-\theta) = f_0 \sin pt$

$= f_0 \sin(pt - \theta + \theta)$

$= f_0 \sin[(pt-\theta) + \theta]$

$= f_0\left[\sin(pt-\theta)\cos\theta + \cos(pt-\theta)\sin\theta\right]$

$= f_0 \sin(pt-\theta)\cos\theta + f_0\cos(pt-\theta)\sin\theta$

Equating the coeff. of $\sin(pt-\theta)$ & $\cos(pt-\theta)$

ie: $f_0 \cos\theta = A(\omega_0^2 - p^2)$ —②

$f_0 \sin\theta = 2KpA$ —③

Sq & add, we get: $②^2 + ③^2$

$$f_0^2 = A^2(\omega_0^2-p^2)^2 + (2KpA)^2$$

$$\cancel{f_0 = \sqrt{A^2\omega_0^2 - p^2}}$$

$$\boxed{A = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4K^2p^2}}}$$ → Amplitude

Phase diff -

$\frac{③}{②} \rightarrow \tan\theta = \frac{2Kp}{(\omega_0^2-p^2)}$

$$\boxed{\theta = \tan^{-1}\frac{2Kp}{(\omega_0^2-p^2)}}$$ → phase diff

Substituting 'A' in PI, $x = A\sin(pt-\theta)$:

$$x = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4K^2p^2}} \sin(pt-\theta)$$

The gen. sol. of FHO is:

$$x = CF + PI$$

$$\boxed{x = a_0 e^{-Kt} \sin(\omega t + \phi) + \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4K^2p^2}} \sin(pt - \theta)}$$

damped   forced

Initially both vibrations are present in FHO, but with the passage of time, first term vanishes.

ie: The sol. of FHO is:

$$\boxed{x = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4K^2p^2}} \sin(pt - \theta)}$$

Resonance condition for FHO

1) Amplitude resonance.

The amplitude of FHO is max. at a particular frequency, which is near to its natural frequency.

Resonant freq. is denoted by P_R

We know that $A = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4K^2p^2}}$

For $A = A_{max}$, denominator is min.

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∴ For denominator to ~~b~~ be min,

$$\frac{d}{dP}(Dr) = 0$$

ie: $\frac{d}{dp}\left[(\omega_0^2 - p^2)^2 + 4K^2p^2\right] = 0$

$$-4p(\omega_0^2 - p^2) + 2K^2p = 0$$

ie: $\cancel{8K^2p = 4p(\omega_0^2 - p^2)^2}$

$$\cancel{2K^2 = (\omega_0^2 - p^2)^2}$$

$$\cancel{\sqrt{2}K = \omega_0^2 - p^2}$$

$$8K^2p = 4p(\omega_0^2 - p^2)$$

$$p^2 = \omega_0^2 - 2K^2$$

When $A = A_{max}$, $P = P_R$

ie: $\boxed{P_R = \sqrt{\omega_0^2 - 2K^2}}$ — ①

Substituting P_R in A_{max},

$$A_{max} = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4K^2p^2}} \quad ; P = P_R$$

$$= \cancel{\frac{f_0}{\sqrt{(\omega_0^2 - \omega_0^2 + 2K^2)^2 + 4K^2(\omega_0^2 - 2K^2)}}}$$

$$=$$

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$$= \frac{f_0}{\sqrt{(\omega_0^2 - p_R^2)^2 + 4K^2 p_R^2}}$$

$(1) \rightarrow p_R^2 = \omega_0^2 - 2K^2 \rightarrow \omega_0^2 - p_R^2 = 2K^2$

$$= \frac{f_0}{\sqrt{(2K^2)^2 + 4K^2 p_R^2}}$$

$$\boxed{A_{max} = \frac{f_0}{2K\sqrt{K^2 + p_R^2}}}$$

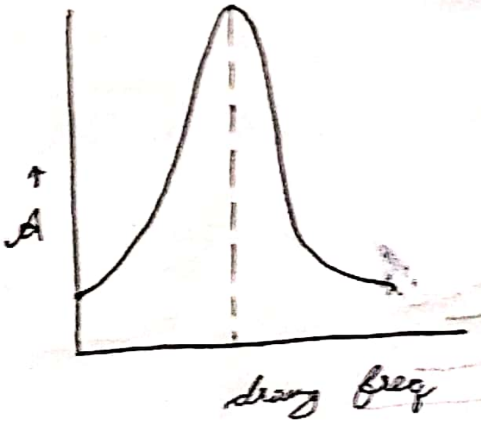
Case 1 (low damping)

Neglecting K^2

$\Rightarrow p_R \approx \omega_0$; $\cancel{A_{max} = \frac{f_0}{2K\omega_0}}$

~~Case 2 (Absence of damping)~~

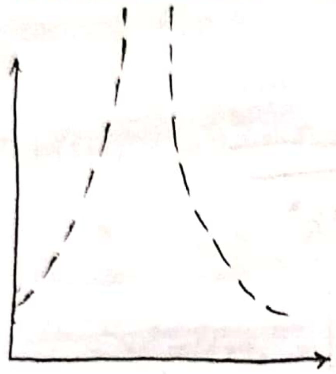
$$\therefore A_{max} = \frac{f_0}{2K\omega_0}$$



Case 2 (Absence/0 damping)

$K = 0$

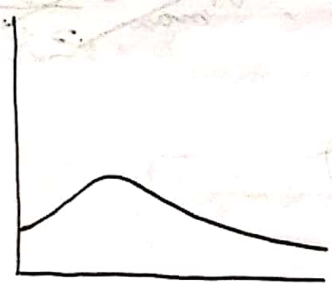
$\Rightarrow A_{resonance} = \infty$



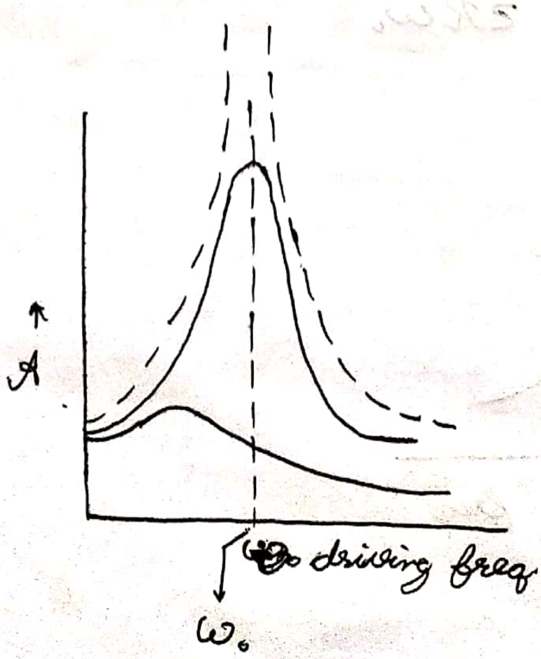
Case 3 (High/finite damping)

A decreases.

ie: $p_R < \omega_0$



ie:



Sharpness & flat of resonance

Amplitude of resonance decreases rapidly on both sides of the resonant freq. This resonance cond. is called sharpness.
eg: sonometer

In the case of low damping

Amplitude decreases very slowly on either sides. This is called flat of resonance.

In the case of high damping

eg: air column exp

Quality factor at resonance ($F \neq 0$)

It is the ratio of amplitude at max to the amplitude at zero driving freq.

ie: $p=0$.

ie: $\boxed{Q = \dfrac{A_{max}}{A_{p=0}}}$

$$A_{max} = \frac{f_0}{2K\omega_0}$$

$$A = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2+4K^2p^2}}$$

; $p=0$

$$= \frac{\dfrac{\cancel{f_0}}{2K\omega_0}}{\dfrac{\cancel{f_0}}{\omega_0^2}}$$

$$\boxed{Q = \frac{\omega_0}{2K} = \omega_0\tau}$$

; $\tau = \dfrac{1}{2K}$

Now: $Q = \dfrac{\omega_0}{2K}$ $\qquad \omega_0 = \sqrt{k/m}$

$K = \dfrac{\gamma}{2m}$

$= \dfrac{\sqrt{k/m}}{\gamma/m}$

$= \dfrac{\sqrt{km}}{\gamma}$

$$\boxed{Q = \frac{\omega_0}{2K} = \omega_0 \tau = \frac{\sqrt{km}}{\gamma}}$$

Q. In the case of forced harmonic oscillator the amplitude increases from 0.02 mm at very low freq. to 5 mm at $\nu = 100$ Hz. Find Q.F, damping const. & relaxation time.

Ans. $A_0 = 0.02$ mm

$A_{max} = 5$ mm

$Q = \dfrac{A_{max}}{A_{p=0}} = \dfrac{5}{0.02} = \underline{\underline{250}}$

$\omega = 2\pi\nu = 200\pi$ rad/s

Relaxation time $\tau = \dfrac{Q}{\omega_0} = \dfrac{250}{200\pi} = \underline{\underline{3.92}}$

Damping const $K = \dfrac{\omega_0}{2Q} = \dfrac{200\pi}{2 \times 250} = \dfrac{200\pi}{500} = \underline{\underline{1.25}}$

Q. A damped vibrating sys. from rest reaches a 1^{st} Amp. of 500 mm, which reduces to 50 mm after 100 osci, each of time period 2.3 sec. Find damp. const & relaxation time.

Ans. $a_0 = e^{-KT}$

$$\frac{\cancel{500}^{10}}{50} = \frac{e^{-KT}}{e^{-K100T}}$$

$$10 = \frac{e^{100KT}}{e^{KT}} = e^{99KT}$$

$$10 = e^{227.7K}$$

$$\ln 10 = 227.7K$$

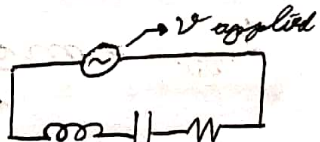
$$2.303 = 227.7K$$

$$\underline{\underline{K = 0.01}}$$

$$\tau = \frac{1}{2K} = \underline{\underline{50}}$$

Comparison of ~~analog~~ mechanical & electric oscillator

$$FHO \rightarrow m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + Kx = F_{applied} \quad -(1)$$



In an LCR circuit;

Pot. diff. across each component is:

Inductor : $V_L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$

Resistor : $V_R = R\frac{dq}{dt}$

Capacitor: $V_c = \frac{q}{C}$

Consider the mechanical oscillator as FHO. Its force eqn. is given above.

For an LCR circuit, the voltage eqn. is given by the sum of P.d across each component.

$$V_{applied} = V_L + V_R + V_C$$

$$V_{appl.} = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q \quad - (2)$$

On comparison of eq. (1) & (2), we can compare the terms.

$$\left.\begin{aligned} \text{ie: } V_{appl} &= L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q \\ F_{appl} &= m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + Kx \end{aligned}\right\}$$

ie: FHO force eq. is similar to voltage eq. of LCR circuit.

Mechanical Oscillator	Electrical Oscillator
Mass (m)	Inductance (L)
Displacement (x)	Charge (q)
Velocity (dx/dt)	Current (dq/dt)
Damping const (K)	$1/C$
Damping coeff (γ)	Resistance (R)
Pot. energy ($\frac{1}{2}Kx^2$)	$\frac{1}{2}Cv^2$
K.E ($\frac{1}{2}mv^2$)	$\frac{1}{2}L\dot{q}^2$
Q.F ; $Q = \frac{\sqrt{Cm}}{\gamma}$	$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{L\omega}{R}$
Resonant freq. $\nu_R = \sqrt{\omega_0^2 - 2K^2}$	$\nu_R = \frac{1}{2\pi\sqrt{LC}}$

Q. Find the natural freq. of a circuit containing inductor of 144 μH & capacitance of 0.0025 μF. To which λ will its response will be max.

Ans $L = 144\,\mu H$; $C = 0.0025\,\mu F$

$$\nu = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{144 \times 0.0025 \times 10^{-12}}} = 2.65 \times 10^{-5}\ Hz$$

$$c = \nu\lambda \Rightarrow \lambda = \frac{c}{\nu}$$

$$= \frac{3\times10^{8}}{2.65\times10^{-5}}$$

$$= \underline{\underline{1.13\times10^{13}\ m}}$$

$$\frac{1}{C\omega} = L\omega$$

$$\omega^2 = \frac{1}{LC}$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$2\pi\nu = \frac{1}{\sqrt{LC}}$$

$$\nu = \frac{1}{2\pi\sqrt{LC}}$$

Q. Deduce freq. & Q.F for a circuit with $L = 2\ mH$, $C = 5\ \mu F$, $R = 0.2\ \Omega$

Ans. $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$

$$= \frac{1}{0.2}\sqrt{\frac{2\times10^{-3}}{5\times10^{-6}}}$$

$$= \frac{1}{0.2}\sqrt{\frac{2}{5}\times10^{3}} = \frac{20}{0.2} = \underline{\underline{100}}$$

$$f = \frac{1}{2\pi\sqrt{LC}} = \underline{1.591\times10^{3}\ Hz}$$

★ Q. In LCR circuit, $L = 10\ mH$, $C = 1\ \mu F$ & $R = 0.1\ \Omega$. How long ~~the~~ does the oscillation take to decay to $\frac{1}{2}$ the amplitude Q.F?

Ans. $A = a_0 e^{-Kt}$

$$\cancel{\frac{A_1 = a_0 e^{-Kt_1}}{\frac{A_1}{2} = a_0 e^{-Kt_2}}}$$

$$\frac{A = a_0 e^{-Kt_1}}{\frac{A}{2} = a_0 e^{-Kt_2}}$$

$$2 = \frac{e^{-Kt_1}}{e^{-Kt_2}}$$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$= \frac{1}{0.1}\sqrt{\frac{10\times10^{-3}}{1\times10^{-6}}}$$

$$= \frac{1}{0.1}\sqrt{10^{4}} = \frac{10^{2}}{0.1}$$

$$= \underline{\underline{1000}}$$

$z = e^{K(t\ldots}$

$\omega = \frac{1}{\sqrt{LC}}$

$Q = \frac{\omega}{2K}$

$\underline{\underline{K = 5}}$

$2 = e\ldots$

$0.693 = \ldots$

Q. Rms volt ... $R = 10\ \Omega$, $L\ldots$ Calc natural ... resonance, ...

Ans. $\omega = \frac{1}{\sqrt{LC}}$

$\nu = \frac{1}{2\pi\sqrt{L\ldots}}$

$= \underline{1592\ldots}$

At resonance ...

$Z = \sqrt{R^2 + (\ldots}$

$$2 = e^{K(t_2 - t_1)}$$

$$2 = e^{\frac{t_2 - t_1}{1\times10^{-6}}}$$

$$0.693 = \frac{t_2 - t_1}{10^{-6}} \Rightarrow T = 0.693\times10^{-6}\,s$$

$$\omega = \frac{1}{\sqrt{LC}} = 10000$$

$Q = \frac{L\omega}{R}$

$$Q = \frac{\omega}{2K} \Rightarrow K = \frac{\omega}{2Q} = \frac{10000}{2\times1000} = \underline{\underline{5}}$$

$\underline{\underline{K = 5}}$

$$2 = e^{KT}$$

$$0.693 = 5T$$

$$\Rightarrow \underline{\underline{T = 0.138\,s}}$$

Q. Rms volt of 100 V is applied to LCR of $R = 10\,\Omega$, $L = 10\times10^{-3}$ H, $C = 1\times10^{-6}$ F. Calc natural freq. of circuit, current at resonance, Q.F, ~~~~ band width.

Ans. $\omega = \frac{1}{\sqrt{LC}}$

$10\times10^{-3}\times10^{-6}$

10^{-8}

$$\nu = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{10^{-8}}}$$

$$= \underline{\underline{1592\ Hz}}$$

At resonance, $X_C = X_L$

$$Z = \sqrt{R^2 + (X_C - X_L)^2} = 10$$

$$i = \frac{100}{10} = 10\,A$$

$$Q.F = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$= \frac{1}{10}\sqrt{\frac{10\times10^{-3}\ 10^{-4}}{10^{-6}}}$$

$$= \frac{1}{10}\times 10^{2} = \underline{\underline{10}}$$

~~B.W~~ $\boxed{\text{Bandwidth} = \frac{\omega}{Q}}$

$$\therefore B.W = \frac{\omega}{Q} = \frac{2\pi\nu}{Q} = \frac{1000.2}{\cancel{10}} = \underline{\underline{1000.2 \text{ rad/sec}}}$$

Q. Amplitude of underdamped harmonic oscillator reduced to 1/10th of initial value after 100 oscillations. Its time period after 100 osci is 1.15 sec. Calc. relaxation time & damping const.

Ans $\cancel{A} = a_0 e^{-Kt}$

$$\frac{a}{10} = a_0 e^{-K100t}$$

$$10 = e^{K(100t - t)}$$

$$= e^{99tK}$$

$$= e^{113.85K}$$

$$2.303 = 113.85K$$

$$\underline{\underline{K = 0.02}}$$

$$\tau = \frac{1}{2K} = \frac{1}{0.04} = \underline{\underline{25}}\text{ s}$$

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Q. Find natural freq. of a circuit containg $L = 50\times10^{-3}$ H, $C = 500\times10^{-12}$ & λ.

Ans. $\omega = \frac{1}{\sqrt{25000\times10^{-12}}} = \frac{1}{5\times10^{-6}}$ $\sqrt{50\times500\times10^{-15}}$

$= 0.2\times10^{6}$

$\nu = \frac{10^5}{\pi} = 31830.98$ Hz $= 3.18\times10^{4}$ Hz.

$c = \nu\lambda \Rightarrow \lambda = \frac{c}{\nu} = 9.424\times10^{3}$ m

Waves

Differential eq. of 1-dim. wave

It is represented as a funct. of pos. & time.

Consider a 1-D wave with vel. 'v', moving in +ve dir.

$$\boxed{\Psi_{(x,t)} = f(x - vt)} \rightarrow +x \text{ dir.} \quad -①$$

$$\Psi_{(x,t)} = f(x + vt) \rightarrow -x \text{ dir.} \quad -②$$

In gen;

$$\boxed{\Psi_{(x,t)} = f(x \pm vt)}$$

Differentiating ① partially wrt. x twice,

$$\frac{\partial \Psi}{\partial x} = f'(x - vt)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = f''(x - vt)$$

Now, wrt. t, twice

$$\frac{\partial^2 \Psi}{\partial t^2} = v^2 f''(x - vt)$$

ie: $\dfrac{\partial^2 \psi}{\partial t^2} = v^2 \dfrac{\partial^2 \psi}{\partial x^2}$ $\left(\because \dfrac{\partial^2 \psi}{\partial x^2} = f''(x - vt)\right)$

$$\therefore \boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}}$$

Its sol. is given by:

$$\boxed{\psi = A \sin[k(x - vt)]}$$

k = propagation vector /const.

Differential eq. of 3-dim wave

$\psi(x, y, z, t)$

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}}$$

$\nabla = \hat{i} \dfrac{\partial}{\partial x} + \hat{j} \dfrac{\partial}{\partial y} + \hat{k} \dfrac{\partial}{\partial z}$ → del

∴ The eqn. becomes:

$$\boxed{\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}}$$

Its sol. is given by:

$$\boxed{\psi = a e^{\pm i(\vec{k} \cdot \vec{r} \pm \omega t)}}$$

$\hat{k} = k_x \hat{i} + k_y \hat{j} + k_z \hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

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Wavelength & timeperiod

Wavelength is defined as the periodic dist. with space periodicity $\left(\frac{2\pi}{k}\right)$

$$\boxed{\lambda = \frac{2\pi}{k}}$$

Timeperiod is the periodic time with time periodicity $\left(\frac{2\pi}{kv}\right)$

$$\boxed{T = \frac{2\pi}{kv}}$$

Relation b/w v, ϑ, λ, & ν

$$T = \frac{2\pi}{k.v}$$

$$kv = \frac{2\pi}{T} \qquad \because \frac{1}{T} = \nu$$

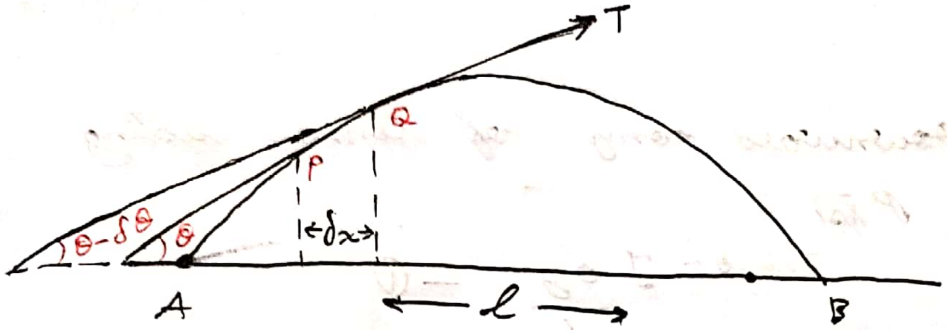
$$= 2\pi\nu$$

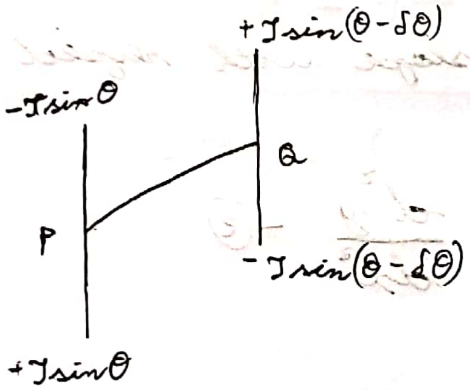
$$v = \frac{2\pi\nu}{k} \qquad ; \lambda = \frac{2\pi}{k}$$

$$= \lambda\nu$$

ie: $\boxed{v = \nu\lambda}$

Transverse vibrations of a stretched string





The downward comp. of tension acting at P.

According to Taylor series expn,

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{5}x^5 + \dots$$

When θ is very small,

$$\sin\theta \approx \tan\theta$$

ie: $T\sin\theta \approx T\tan\theta$

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Slope at P, $\tan\theta$;

$$\tan\theta = \frac{dy}{dx}$$

$\therefore$ Downward comp. of tension acting at P is;

$$T\sin\theta = T\frac{dy}{dx} \quad -① $$

Rate of change of slope with respect to length;

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \quad -②$$

> ~~Rate of~~ Change of slope at a dist. δx is:

$$\cancel{\frac{d^2y}{dx^2}(\delta x)} \quad \frac{d^2y}{dx^2}\times\delta x \quad -③$$

Slope of Q; $\rightarrow$ (slope of P - change in slope at Q).

$$\tan(\theta-\delta\theta) = \frac{dy}{dx} - \frac{d^2y}{dx^2}\delta x \quad -④$$

Upward comp. of T acting at Q is;

$$T\sin(\theta-\delta\theta) \approx T\tan(\theta-\delta\theta)$$

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$$= T\left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\delta x\right) \quad - (5)$$

The resultant downward tension acting at PQ is:

$$\cancel{T\frac{dy}{dx} - \left[T\frac{dy}{dx}\right.}$$

$$T\sin\theta - T\sin(\theta - \delta\theta)$$

$$= T\frac{dy}{dx} - \left[T\left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\delta x\right)\right]$$

$$= T\frac{d^2y}{dx^2}\delta x \quad - (6)$$

Consider 'm' is the mass per unit length of the string.

⇒ Mass of the small elem. δx is

$m\delta x$.

Force acting on the element.

$$= m\delta x\, a$$

$$F = m\delta x\frac{d^2y}{dt^2} \quad - (7)$$

Comparing (6) & (7), we get;

$$m\cancel{\delta x}\frac{d^2y}{dt^2} = T\frac{d^2y}{dx^2}\cancel{\delta x}$$

$$\boxed{\frac{d^2y}{dt^2} = \frac{T}{m}\frac{d^2y}{dx^2}} \quad -⑧$$

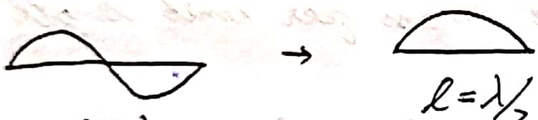
similar to 1-D wave eqn;

$$\boxed{\frac{d^2\psi}{dt^2} = v^2\frac{d^2\psi}{dx^2}} \quad -⑨ .$$

Comparing ⑧ & ⑨

$$v^2 = \frac{T}{m} \Rightarrow \boxed{v = \sqrt{\frac{T}{m}}}$$ → vel. of propagation

$$\nu = \frac{v}{\lambda} = \frac{1}{\lambda}\sqrt{\frac{T}{m}}$$



$l = \lambda$ → $l = \lambda/2$

If the string is vibrating in one segment;
$l = \lambda/2$

1 mode

$$\Rightarrow \boxed{\nu = \frac{1}{2l}\sqrt{\frac{T}{m}}}$$

If no. of modes is given as P;

$$\boxed{\nu = \frac{P}{2l}\sqrt{\frac{T}{m}}}$$

Q. A flexible string of $l = 0.8$ m & $M = 2.5$ gm is stretched to produce vibrations: vibrates in 4 segments with $\nu = 600$ Hz. Calc. T

$1g = 10$

Ans $l = 0.8$ m

$M = 2.5$ gm

$m = \frac{2.5}{0.8} = 3.125 \cancel{mg/m}\ gm/mm = 3.125 \times 10^{-3}$ kg/m

$p = 4$ nodes.

$\nu = 600$ Hz.

$$\nu = \frac{p}{2l}\sqrt{\frac{T}{m}}$$

$$T = \frac{\nu^2 4l^2}{p^2} \times m = 180\ N$$

Q. A string of violin $l = 36$ cm & $M = 0.2$ gm. With what tension it must be stretched to tune 1000 Hz.

Ans. $l = 36 \times 10^{-2}$ m

$M = 0.2 \times 10^{-3}$ kg

$m = \frac{0.2 \times 10^{-3} 10^{-2}}{36 \times 10^{-2}} = \frac{2}{36} \times 10^{-2} = 0.055 \times 10^{-2}$ kg/m

$\nu = 1000$ Hz.

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{m}} \Rightarrow$$

$$T = \nu^2 4l^2 m = 288.2\ N$$

Q. Calc. freq. of fundamental note produced by string 1 m long & weighing 2 gm. stretched by load 400 kg.

Ans. $\nu = ?$

$l = 1\ m$

$M = 2\times10^{-3}\ kg$

$m = 2\times10^{-3}\ kg/m$

$T = 400\ kg \times 9.8\ m/s^2$

$= \underline{3920\ N}$

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{m}} = \frac{1}{2}\sqrt{\frac{3920}{2\times10^{-3}}}$$

$$= \underline{\underline{700\ Hz}}$$

Q. Find ν, T & $\bar{\nu}$ of $\lambda = 800\ nm$.

$c = \nu\lambda$

$$\nu = \frac{c}{\lambda} = \frac{3\times10^{8}}{800\times10^{-9}} = \cancel{375000\ m}$$

$$= \underline{\underline{3.75\times10^{14}\ Hz}}$$

$$\bar{\nu} = \frac{1}{\lambda} = \frac{1}{800\times10^{-9}} = \underline{\underline{1250000}}\ m^{-1}$$

$$T = \frac{1}{\nu} = \underline{\underline{2.66\times10^{-15}\ s}}$$

Q. Calc. propagation const. & angular freq. of $\lambda = 700$ nm.

Ans $\lambda = \frac{2\pi}{k}$

$k = \frac{2\pi}{\lambda} = 8.97 \times 10^{-12}$

$c = \nu\lambda \Rightarrow \nu = \frac{c}{\lambda} = \frac{3 \times 10^{8}}{7 \times 10^{-7}}$

$10^{-9} \times 10^{2}$

$$\begin{array}{r} 0.42 \\ 7 \enclose{longdiv}{30} \\ 28 \\ \hline 20 \end{array}$$

$= 0.42 \times 10^{15}$

$= \underline{\underline{4.2 \times 10^{14} \text{ Hz}}}$

$\omega = 2\pi\nu$

$= \underline{\underline{2.63 \times 10^{15}}}$